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## The Three Loop Equation of State of QED at High Temperature

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### Abstract

We present the three loop contribution (order  $e^4$ ) to the pressure of massless quantum electrodynamics at nonzero temperature. The calculation is performed within the imaginary time formalism. Dimensional regularization is used to handle the usual, intermediate stage, ultraviolet and infrared singularities, and also to prevent overcounting of diagrams during resummation.

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The equation of state (EOS) of relativistic quantum electrodynamics (QED) at nonzero temperature ( $T$ ) and chemical potential ( $\mu$ ) is of relevance in several astrophysical contexts [1, 2, 3, 4]. It was obtained by Akhiezer and Peletminskii to the third order ( $e^3$ ) more than three decades ago while the fourth order ( $e^4$ ) contribution at  $T = 0$ , but nonzero  $\mu$ , was then obtained for QED and quantum chromodynamics (QCD) by Freedman and McLerran [2], and Baluni [3]. However, to date, the analogous 3-loop calculation for  $T \neq 0$  has, to our knowledge, not been performed, presumably because of the greater technical difficulty in dealing with overlapping diagrams in the presence of Bose-Einstein and Fermi-Dirac statistical factors.

In this Letter we present the order  $e^4$  contribution to the pressure of a QED plasma at  $T \neq 0$  (and  $\mu = 0$ ), but for the case of massless electrons. This will also be the leading contribution at high temperature for realistic QED with massive electrons.

Our motivation for this high order calculation is two-fold. Firstly, for phenomenological applications it is helpful to know how big corrections to the lower order EOS can be. Secondly, it serves as a prototype to illustrate techniques which may be used to perform similar calculations in QCD. Recall that in QCD asymptotic freedom suggests [5] that at high  $T$  and/or density hadronic matter will transform into a weakly interacting quark-gluon plasma, a novel state of matter that is currently under intense study [6]. The EOS for the high-temperature phase of QCD was determined by Kapusta to third order ( $g^3$ ) [7] while Toimela [8] has extracted also the  $g^4 \ln g$  piece. However, the normalization of the logarithm in the last term requires knowledge of the 3-loop contribution which is still lacking.

Returning to QED, let us introduce our notation and conventions. We employ the imaginary time formalism whereby the energies take on discrete Matsubara values,  $q_0 = in\pi T$ ,  $n$  being an even (odd) integer for bosons (fermions). Dimensional regularisation is used to handle the ultraviolet (UV) and infrared (IR) singularities which occur at intermediate steps. The  $D$  dimensional vector,  $Q_\mu = (q_0, \vec{q})$  is contracted with a Minkowski metric,  $Q^2 = q_0^2 - \vec{q}^2$ . In order to keep track of the even (odd)

Matsubara frequencies, we introduce the following notation:

$$\int [dq] \equiv T \sum_{q_0, \text{even}} \int \frac{d^{D-1}q}{(2\pi)^{D-1}},$$

$$\int \{dq\} \equiv T \sum_{q_0, \text{odd}} \int \frac{d^{D-1}q}{(2\pi)^{D-1}}.$$

The fermions are kept as four-component objects,  $Tr(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu}$ , and for simplicity we will work in the Feynman gauge so that the gauge propagator is  $g_{\mu\nu}/K^2$ . Renormalization via minimal subtraction ensures that the coupling constant is gauge-fixing independent, and hence so will then be our final answer for the pressure [3].

Before discussing the 3-loop calculation, let us summarize the lower order results for the pressure of QED with  $N$  massless Dirac fermions :

$$P = P_0 + P_2 + P_3 + O(e^4), \quad (1)$$

where

$$P_0 = \frac{\pi^2}{45} T^4 \left(1 + \frac{7}{4} N\right), \quad (2)$$

$$P_2 = -\frac{5e^2 T^4 N}{288}, \quad (3)$$

$$P_3 = \frac{e^3 T^4}{12\pi} \left(\frac{N}{3}\right)^{3/2}. \quad (4)$$

The ideal gas contribution  $P_0$  is determined by the one-loop diagrams in Fig. 1. The first correction,  $P_2$ , is given by Fig. 2. The nonanalytic  $(e^2)^{3/2}$  contribution,  $P_3$ , is a consequence of Debye screening [9]. In a perturbative expansion using bare propagators one discovers power-like IR singularities in diagrams such as Fig. 3, corresponding to the  $n = 0$  Matsubara frequency of the photon propagator. The diagrams in Fig. 3 are singular because the electric polarization operator behaves as

$\Pi_{00}(q_0, q \rightarrow 0) = m^2$ , where  $m$  is the electric screening mass to lowest order. Summing the infrared divergent pieces of the diagrams in Fig. 3 yields

$$P_3 = \frac{T}{2} \sum_{p=2}^{\infty} \int \frac{d^3 q}{(2\pi)^3} \frac{(-1)^p}{p} \left( \frac{m^2}{q^2} \right)^p \quad (5)$$

$$= -\frac{T}{2} \int \frac{d^3 q}{(2\pi)^3} \left( \ln\left(1 + \frac{m^2}{q^2}\right) - \frac{m^2}{q^2} \right). \quad (6)$$

Although (6) is UV and IR finite and may be evaluated directly to give (4), it will be helpful to reconsider it using dimensional continuation. Then [10] the second term in (6) vanishes and the first term gives

$$\frac{T}{2} \Gamma\left(\frac{1-D}{2}\right) \left(\frac{m^2}{4\pi}\right)^{(D-1)/2},$$

which is  $P_3$  as  $D \rightarrow 4$ . Note that in dimensional regularization (DR) each single integral of the series in (5) vanishes. Nevertheless the result in (6) is physically correct, and mathematically nonzero, because the sum in (5) must be done *before* the integral. This is obvious once one starts from an expansion of the pressure in terms in terms of the full propagator [1, 2, 3]. Though the infinite sum must be performed first, we may separate out any *finite* number of terms from the sum, and for these we may use DR to deduce a zero contribution. This explains why the second term in (6) may be dropped in DR and we will exploit this fact further below.

Consider now the 3-loop diagrams. The order  $e^4 N$  diagrams are shown in Fig. 4 and 5. However, for massless fermions, the Ward identity  $Z_1 = Z_2$  implies the mutual cancellation of the counterterm diagrams. Thus the sum  $G_1 + G_2$  (Fig. 4) is UV finite. After performing the spinor traces, some algebraic manipulation, and the use of scaling arguments such as in [11], we obtain

$$\begin{aligned} \frac{G_1 + G_2}{e^4 N T^{3D-8} \mu^{8-2D}} &= \frac{(D-2)}{6} \left( 2 (1 - 2^{(11-3D)}) H_1 + (20 - 3D) H_2 \right) \\ &+ (D-2)^2 ( 2 H_3 - f_2 (f_1 - b_1)^2 ). \end{aligned}$$

Here  $\mu$  is the mass parameter of DR and  $e$  is the dimensionless, renormalized coupling. We have scaled all the momenta by  $1/T$  so that the integrals are dimensionless (i.e.

$T = 1$  there) and are defined by

$$b_n \equiv \int \frac{[dQ]}{(Q^2)^n},$$

$$f_n \equiv \int \frac{\{dQ\}}{(Q^2)^n},$$

$$H_1 = \int \frac{[dQ dP dK]}{K^2 Q^2 P^2 (K + Q + P)^2},$$

$$H_2 = \int \frac{\{dK dR dS\}}{K^2 R^2 S^2 (K + R + S)^2},$$

$$H_3 = \int \frac{\{dK\} [dQ dP] (P \cdot Q)}{K^2 P^2 Q^2 (K + Q)^2 (K + P)^2}. \quad (7)$$

The integral  $H_1$  occurs in the 3-loop evaluation of the pressure in  $\phi^4$  theory [12], and  $H_2$ , being simply the fermionic analog of  $H_1$ , may be analysed in a similar manner. The only new integral left is  $H_3$ . Let us however first discuss the order  $e^4 N^2$  diagrams. The UV singularity of  $G_3$  (Fig. 6a) is cancelled by the photon wave-function renormalization through diagram,  $X_1$  (Fig. 6b). Diagram  $G_3$  also has an IR singularity which is precisely the first term of the series in eq.(5). Since this term has already been considered there, it should be subtracted from  $G_3$  to avoid overcounting. However, as discussed earlier, this single term by itself vanishes in DR, so double-counting is automatically avoided. We have

$$G_3 = \frac{e^4 N^2}{4} T^{3D-8} \mu^{8-2D} 16 \left( (D-4) b_2 f_1^2 + \frac{(D-4)}{4} H_2 + 4 H_4 \right), \quad (8)$$

and

$$X_1 = -(Z_3 - 1) e^2 N (D-2) T^{3D-8} \mu^{8-2D} f_1 (2b_1 - f_1) \left( \frac{T}{\mu} \right)^{4-D},$$

where

$$Z_3 - 1 = \frac{e^2 N}{6\pi^2(D-4)}$$

to leading order. The factor  $\left(\frac{T}{\mu}\right)^{4-D}$  will generate a  $e^4 \ln T/\mu$  term on expansion, but this will be reabsorbed later in the definition of the temperature dependent coupling  $e(T)$ . The new integral in (8) is

$$H_4 = \int \frac{[dQ]\{dK dR\} (K \cdot R)^2}{Q^4 K^2 R^2 (Q+K)^2 (Q+R)^2}.$$

We now sketch our evaluation of  $H_3$  (7). The frequency sums are first rewritten in terms of contour-integrals as in [2, 3] but we do not separate the  $T = 0$  and  $T \neq 0$  parts before going to the “phase-space” representation. In this way we obtain

$$H_3 = J_1 + K_1 + L_1.$$

The piece  $L_1$  contains integrals which can be performed analytically while  $K_1$  is an integral similar to  $H_1$ . The difficulty lies in

$$J_1 \equiv \int \frac{d^D K d^D Q d^D P}{(2\pi)^{3(D-1)}} \delta_+(K^2) \delta_+(P^2) \delta_+(Q^2) N_{k_0} n_{q_0} (N_{p_0} + n_{p_0}) \sum_{\sigma, \gamma = \pm 1} (-\sigma) S(\sigma, \gamma),$$

with

$$S(\sigma, \gamma) \equiv \frac{P \cdot Q}{K \cdot Q} \frac{1}{K \cdot Q + P \cdot (\sigma K + \gamma Q)},$$

and

$$N_{k_0} = \frac{1}{e^{k_0} + 1}.$$

The integral  $J_1$  is UV finite but has a collinear singularity. We use the method of Sudakov decomposition, a technique which is well known at  $T = 0$  (see for example [13]) but appears to be novel in this context, to extract the pole and the finite part

of this integral. The details are lengthy and will be presented elsewhere [14]. Here we only state the final result,

$$J_1 = \frac{1}{128\pi^4} \left( \frac{r_1}{(D-4)} + c_1 \right),$$

where

$$\begin{aligned} r_1 &= -0.7167667897 \pm 10^{-10}, \\ c_1 &= 3.936 \pm 5 \times 10^{-3}. \end{aligned}$$

The residue  $r_1$  is obtained as a finite two-dimensional integral which we have then evaluated numerically to high precision (relative error  $10^{-10}$ ), and the pole in  $J_1$  above cancels with the other poles contributing to the sum of  $G_1 + G_2$  (again to the same precision). For  $H_4$ , the analysis is similar to  $H_3$  but more tedious because of the doubled propagator  $1/(Q^2)^2$ .

Collecting all the pieces, we obtain

$$\begin{aligned} \frac{P_4}{e^4 T^4} &= \frac{(G_1 + G_2 + G_3 + X_1)}{e^4 T^4} \\ &= \frac{N}{\pi^6} (0.4056) - N^2 \left( \frac{0.4667}{\pi^6} + \frac{5}{6\pi^2 \times 288} \ln \frac{T}{\mu} \right). \end{aligned} \quad (9)$$

The pressure up to and including order  $e^4$  then follows from eqs.(1-4) and (9). It may be rewritten in terms of the (one-loop) renormalization group invariant coupling, at the energy scale  $T$ , given by

$$e^2(T) = e^2 \left( 1 + \frac{e^2 N}{6\pi^2} \ln \frac{T}{\mu} \right).$$

Defining  $\alpha(T) = e^2(T)/4\pi$  we finally arrive at

$$\begin{aligned} \frac{P}{T^4} &= \frac{\pi^2}{45} \left( 1 + \frac{7}{4} N \right) - \frac{5\pi^2}{72} \frac{\alpha(T)N}{\pi} + \frac{2\pi^2}{9\sqrt{3}} \left( \frac{\alpha(T)N}{\pi} \right)^{3/2} \\ &\quad + \left( \frac{0.658 \pm 0.006}{N} - 0.757 \pm 0.004 \right) \left( \frac{\alpha(T)N}{\pi} \right)^2 + O(\alpha(T)^{5/2}). \end{aligned} \quad (10)$$

This (10) is our expression for the three-loop pressure of QED with  $N$  electron flavours at high temperature; if  $m_e$  is the (zero temperature) electron mass, we require  $m_e/T \ll \alpha(T)^2$  so that mass corrections are subleading to the terms displayed in (10). Real world QED corresponds to  $N = 1$  and in the regime where  $\alpha(T) \ll 1$ , the three loop contribution is found to be indeed a small correction. However, one should note that since perturbative QED is not asymptotically free, the effective coupling  $\alpha(T)$  increases with temperature (albeit slowly) so that at sufficiently high temperatures the fourth order contribution becomes relevant. Deferring further discussion and potential applications of (10) to a later stage [14], we mention that the unknown  $e^5$  contribution in eq.(10) is a higher order analog of the  $e^3$  plasmon term and is also calculable [15].

We conclude by summarizing the steps leading from the Feynman diagrams to the result (10) : (i) algebraic reduction of the integrals, (ii) evaluation of frequency sums by a contour-integral algorithm, (iii) evaluation of the final phase-space-like integrals, in particular using the method of Sudakov variables for UV finite integrals with a collinear singularity, and, (iv) use of dimensional regularization to simplify the prevention of double counting during resummation. We hope that the methodology adopted here opens the way for a similar, and long awaited, calculation in QCD.

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## References

- [1] I.A. Akhiezer and S.V. Peletminskii, Sov. Phys. JETP 11, 1316 (1960).
- [2] B.A. Freedman and L.D. McLerran, Phys. Rev. D16, 1130, 1147, 1169 (1977).
- [3] V. Baluni, Phys. Rev. D17, 2092 (1978).
- [4] A. Heckler, Phys. Rev. D49, 611 (1994), and references therein.
- [5] J.C Collins and M.J. Perry, Phys. Rev. Lett. 34, 1353 (1975).
- [6] See, for example, “Hot Summer Daze”, BNL summer study on QCD at Nonzero Temperature and Density; BNL, NY, Aug. 7-16, 1991; A.Gocksch and R. Pisarski eds. (World Scientific, Singapore, 1992).
- [7] J.I. Kapusta, Nucl. Phys. B148, 461 (1979).
- [8] T. Toimela, Phys. Lett. 124B, 407 (1983).
- [9] D. Bohm and D. Pines, Phys. Rev. 92, 609 (1953);  
M. Gell-Mann and K.A. Brueckner, Phys. Rev. 106, 364 (1957).
- [10] R. Parwani, Phys. Rev. D48, 3852 (1993).
- [11] P. Arnold and O. Espinoza, Phys. Rev. D47, 3546 (1993).
- [12] J. Frenkel, A.V. Saa and J.C. Taylor, Phys. Rev. D46, 3670 (1992).
- [13] G. Stermann, “An Introduction to Quantum Field Theory”,  
(Cambridge Univ. Press, 1993.)  
Yu.L. Dokshitzer, D.I. Dyakonov and S.I. Troyan. Phys. Rep. 58, 270 (1980).
- [14] R. Parwani and C. Corianò, manuscript in preparation.
- [15] R. Parwani, preprint SPhT/94-065.
- [16] P. Arnold and C. Zhai, Seattle preprint UW/PT-94-03.

## Figure Captions

### Fig.1:

Contribution to the ideal gas pressure. The wavy line represents the photon propagator.

### Fig.2:

The two loop diagram.

### Fig.3:

Diagrams contributing to the  $e^3$  plasmon term. The self-energy insertions are  $\Pi_{00}(0, 0)$  and the photon is static,  $q_0 = 0$ .

### Fig.4:

The order  $e^4 N$  contributions :  $G_1$  and  $G_2$ .

### Fig.5:

Ultraviolet counterterm diagrams for Fig.4.

### Fig.6:

Fig.6a is the  $e^4 N^2$  contribution ( $G_3$ ) while Fig.6b is the corresponding counterterm diagram  $X_1$ .

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